

A NOTE ON OPEN 3-MANIFOLDS SUPPORTING FOLIATIONS BY PLANES

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ABSTRACT. We show that if N , an open connected n -manifold with finitely generated fundamental group, is C^2 foliated by closed planes, then $\pi_1(N)$ is a free group. This implies that if $\pi_1(N)$ has an Abelian subgroup of rank greater than one, then \mathcal{F} has at least a non closed leaf. Next, we show that if N is three dimensional with fundamental group abelian of rank greater than one, then N is homeomorphic to $\mathbb{T}^2 \times \mathbb{R}$. Furthermore, in this case we give a complete description of the foliation.

1. INTRODUCTION

Two foliated manifolds (N_1, \mathcal{F}_1) and (N_2, \mathcal{F}_2) are said to be C^r *conjugate* if there exists a C^r homeomorphism $h : N_1 \rightarrow N_2$ that takes leaves of \mathcal{F}_1 onto leaves of \mathcal{F}_2 . Let \mathcal{F} be a C^2 foliation defined on an n -manifold N . If all the leaves are diffeomorphic to \mathbb{R}^{n-1} the foliation \mathcal{F} is called a *foliation by planes*. \mathcal{F} is called a *Reeb foliation* of N if both N and \mathcal{F} are orientable, each leaf of \mathcal{F} in the interior of N is homeomorphic to \mathbb{R}^{n-1} and if $\partial N \neq \emptyset$, then each component of ∂N is a leaf of \mathcal{F} is homeomorphic to the torus \mathbb{T}^{n-1} . Note that, each transversally orientable foliation by planes \mathcal{F} on an orientable open n -manifold is a Reeb foliation.

Rosenberg–Roussarie in [9] have proved that the only compact connected 3-manifolds which admit Reeb foliations are \mathbb{T}^3 , $D^2 \times \mathbb{S}^1$ and $\mathbb{T}^2 \times [0, 1]$. Novikov [6] has proved that all Reeb foliations of $D^2 \times \mathbb{S}^1$ are topologically equivalent to the Reeb component. Chatelet–Rosenberg [2] and Rosenberg–Roussarie [8] have classified Reeb foliations of class C^2 of $\mathbb{T}^2 \times [0, 1]$ and \mathbb{T}^3 , respectively. In both cases, the authors prove, between other things, that all the leaves of \mathcal{F} are dense (in particular there is no closed leaf). Let us observe that in these two cases the fundamental group of N is abelian of rank greater than one. In higher

Date: May 27, 2009.

2000 Mathematics Subject Classification. 37C85, 57R30.

Key words and phrases. Foliation by planes, Open manifolds, Fundamental group, Free group.

The first author was supported by FAPESP of Brazil Grant 98/13400-5.

The second author was supported by FAPESP of Brazil Grant 02/09425-0.

dimension, that is when N is closed and n dimensional, Rosenberg [11] proved that $\pi_1(N)$ is free abelian. Recently, Álvarez López-Arraut-Biasi [1] proved that if $N \setminus \{p_1, \dots, p_k\}$ is foliated by closed planes, then $N = S^n$ (the n -sphere) and $k = 1$.

Palmeira [7] studied transversally orientable C^2 foliations by closed planes on open n -manifolds N , which has finitely generated fundamental group, and proved that this foliations are C^2 conjugates to the product of a foliation on an open surface by \mathbb{R}^{n-2} . This implies that the fundamental group of N is free. Remember that any free group is the free product $H_1 * \dots * H_\ell$ where H_i is isomorphic to $\{0\}$ or \mathbb{Z} .

In this paper we consider foliations by planes on an open and connected n -manifold N . We try to initiate the classification of some C^2 foliations by planes in open manifolds having at least one leaf not closed.

Our first result is the following

Theorem 1. *Let N be an open connected n -manifold with a C^2 foliation by closed planes. If $\pi_1(N)$ is finitely generated, then $\pi_1(N)$ is free.*

Note that in Theorem 1 it is not necessary to assume that N is orientable and that the foliation is transversally orientable. As a consequence of Theorem 1, we have

Corollary 1. *Let N be an open connected n -manifold with a C^2 foliation by planes. If $\pi_1(N)$ has an Abelian subgroup of rank greater than one, then \mathcal{F} has at least a leaf which is not closed.*

Obviously, the reciprocal of this result is not true, see (2) of Example 1. However, the previous corollary motivates the following natural question: “if $\pi_1(N)$ has an Abelian subgroup (or is Abelian) of rank greater than one, then what we can say on the set of non closed leaves?” In this direction, when N is three dimensional, we obtain the following result:

Theorem 2. *Let N be an open connected orientable 3-manifold with a C^2 foliation by planes \mathcal{F} and assume that $\pi_1(N)$ is finitely generated. Then*

- (1) $\pi_1(N) = H_1 * \dots * H_\ell$, where H_j , $j = 1, \dots, \ell$, is a subgroups of $\pi_1(N)$ isomorphic to \mathbb{Z} or \mathbb{Z}^2 .
- (2) For each $j = 1, \dots, \ell$, satisfying that H_j is isomorphic to \mathbb{Z}^2 , there exists an open submanifold N_j of N that is invariant by \mathcal{F} , such that $\pi_1(N_j) = H_j$ and $(\mathcal{F}|_{N_j}, N_j)$ is topologically conjugated to $(\mathcal{F}_0 \times \mathbb{R}, \mathbb{T}^2 \times \mathbb{R})$, where \mathcal{F}_0 is the foliation on the 2-torus \mathbb{T}^2 which is defined by the irrational flow.

By using this theorem we obtain.

Theorem 3. *Let N be an open connected orientable 3-manifold with a C^2 foliation by planes \mathcal{F} and assume that $\pi_1(N)$ is finitely generated. If $\pi_1(N)$ is an abelian group of rank two, then N is homeomorphic to $\mathbb{T}^2 \times \mathbb{R}$. Moreover, there exists an open submanifold N_0 which is homeomorphic to $\mathbb{T}^2 \times \mathbb{R}$ and invariant by \mathcal{F} , such that*

- (1) *$(\mathcal{F}|_{N_0}, N_0)$ is topologically conjugated to $(\mathcal{F}_0 \times \mathbb{R}, \mathbb{T}^2 \times \mathbb{R})$, where \mathcal{F}_0 is the foliation on the 2-torus defined by the irrational flow.*
- (2) *Every connected component B of $N \setminus \overline{N_0}$ is foliated by closed planes, hence, homeomorphic to \mathbb{R}^3 . Furthermore, each leaf L in B separates N in two connected components.*

We observe that in the hypothesis of Theorem 3, by Proposition 2, we might have supposed that $\pi_1(N)$ has an Abelian subgroup of rank greater than one.

The proof of Theorem 2 is typical of the three dimensional case, since we use a result of [3] on the existence of incompressible torus in irreducible 3-manifolds.

This paper is organized as follows. In Section 2 we present some examples of foliations by planes in open manifolds having leaves not closed. In Section 3, by using two remarkable results obtained by Palmeira (Theorem 4) and Stallings (Lemma 1), we prove the Theorem 1. In Section 4 we prove the Theorem 2 and then by using this theorem we show Theorem 3.

2. EXAMPLES

Let us give some examples of open manifolds supporting foliations by planes having not closed leaves.

Example 1. Let X be the vector field on the 2-torus \mathbb{T}^2 which is defined by the irrational flow. Fix a point p in \mathbb{T}^2 and let $f : \mathbb{T}^2 \rightarrow \mathbb{R}$ an application such that $f(p) = 0$ and $f(x) \neq 0$ for all $x \in \mathbb{T}^2 - \{p\}$. Let \mathcal{F}_0 be the one dimensional foliation in $\mathbb{T}^2 - \{p\}$ which is defined by the vector field fX . We obtain two foliations:

- (1) The foliation by planes \mathcal{F}_1 on $N = \mathbb{T}^2 \times \mathbb{R}^{n-2}$ whose leaves are the product of the orbits of X with \mathbb{R}^{n-2} .
- (2) The foliation by planes \mathcal{F}_2 on $N = (\mathbb{T}^2 - \{p\}) \times \mathbb{R}^{n-2}$, whose leaves are the product of leaves of \mathcal{F}_0 with \mathbb{R}^{n-2} .

We can observe that every leaf in the two foliations \mathcal{F}_1 and \mathcal{F}_2 are dense, in particular, are not closed. But, in the first case the fundamental group of N is isomorphic to \mathbb{Z}^2 and, in the second case the fundamental group of N is free not abelian with two generators (because $\pi_1(N) = \pi_1(\mathbb{T}^2 - \{p\})$).

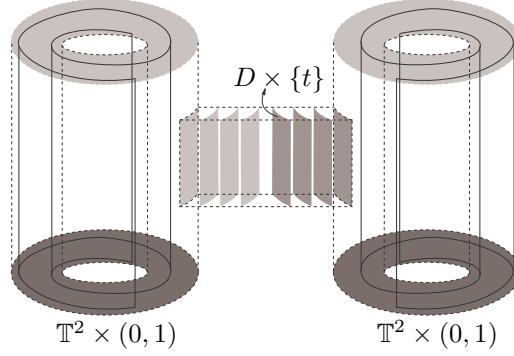


FIGURE 1. Example 2

Example 2. Let \mathcal{F}_0 be the foliation on $\mathbb{T}^2 \times [0, 1]$ such that $T_0 = \mathbb{T}^2 \times \{0\}$ and $T_1 = \mathbb{T}^2 \times \{1\}$ are compact leaves and the restriction $\mathcal{F}_0|_{\mathbb{T}^2 \times (0, 1)}$ is by planes (in fact, this foliation can be defined by a locally free action of \mathbb{R}^2 , see for example page 2 in [2]). In this situation, it is well known that, every leaf of \mathcal{F}_0 in $\mathbb{T}^2 \times (0, 1)$ is dense. Let D be an open 2-disk in T_1 and we consider in $D \times [0, 1]$ the foliation \mathcal{F}_d whose leaves are $D \times \{t\}$ with $t \in [0, 1]$. Let N be the open 3-manifold which is obtained by gluing two copies of $[\mathbb{T}^2 \times (0, 1)] \cup D$ with $D \times I$ as in Figure 2. Let \mathcal{F} be the foliation in N such that $\mathcal{F}|_{\mathbb{T}^2 \times (0, 1)} = \mathcal{F}_0|_{\mathbb{T}^2 \times (0, 1)}$ and $\mathcal{F}|_{D \times I} = \mathcal{F}_d$. Then we have the following facts:

- (1) the fundamental group of N is the free product $\mathbb{Z}^2 * \mathbb{Z}^2$.
- (2) in each copy of $\mathbb{T}^2 \times (0, 1)$ we can find two incompressible 2-tori T_1 and T_2 , which are transverse to \mathcal{F} , such that the foliations induced by \mathcal{F} in this torus are topologically equivalent to irrational flow on the 2-torus. isotopic to $\mathbb{T}^2 \times \{1/2\}$
- (3) for $i = 1, 2$, let $\text{Sat}(T_i)$ be the saturated by \mathcal{F} of T_i . Then, $\text{Sat}(T_i) = \mathbb{T}^2 \times (0, 1)$ and the complement of the closure of $\text{Sat}(T_1) \cup \text{Sat}(T_2)$ is $D \times (0, 1)$ and foliated by closed planes.

Example 3. For $k \geq 2$, let \mathcal{F}_0 be the $(k-1)$ -dimensional foliation by planes in the k -torus \mathbb{T}^k . In $N = \mathbb{T}^k \times \mathbb{R}^{n-k}$, we consider the foliation by planes \mathcal{F} which is the product of \mathcal{F}_0 with \mathbb{R}^{n-k} . Then:

- $\pi_1(N) = \pi_1(\mathbb{T}^k) = \mathbb{Z}^k$ (in particular, Abelian free)
- All the leaves of \mathcal{F} are dense (in particular, this are not closed). This, because that every leaf of \mathcal{F}_0 is dense in \mathbb{T}^k .
- N is not homeomorphic to the product of a surface by \mathbb{R}^{n-2}
- $\text{rank}(\pi_1(N)) = k \geq 2$

3. THE FUNDAMENTAL GROUP OF AN OPEN n -MANIFOLD FOLIATED BY CLOSED PLANES

3.1. Proof of Theorem 1. We state the results to be used in the proof of Theorem 1. We start with the following theorem proved by Palmeira in [7].

Theorem 4. *If N is an orientable open n -manifold, $n \geq 3$, which has a finitely generated fundamental group and with a transversely orientable C^2 foliation \mathcal{F} by closed planes, then there exists an orientable surface Σ and an orientable one-dimensional foliation \mathcal{F}_0 of Σ such that (N, \mathcal{F}) is conjugated by a diffeomorphism to $(\Sigma \times \mathbb{R}^{n-2}, \mathcal{F}_0 \times \mathbb{R}^{n-2})$. When N is simply connected it is not necessary to assume either that \mathcal{F} is transversely orientable or that the leaves are closed, and moreover $\Sigma = \mathbb{R}^2$ in this case.*

Corollary 2. *N have the same homotopy type of an open surface.*

Theorem 4 and the following result, which was conjectured by Serre [12] and proved by Stallings [13], are crucial to prove Theorem 1.

Lemma 1 ([13], Stallings). *Let G be a torsion-free finitely generated group. If H is a free subgroup of finite index, then G is also free.*

Proposition 1. *If N is an open connected n -manifold foliated by planes, then $\pi_1(N)$ is torsion-free. In particular, if $\pi_1(N)$ is finite, then N is diffeomorphic to \mathbb{R}^n .*

Proof. To prove that $\pi_1(N)$ is torsion-free, it is enough to show that its only finite subgroup is the trivial one. Let \mathcal{F} denote the foliation of N by planes and let H be a finite subgroup of $\pi_1(N)$ com k elements. Let $\tilde{N} \rightarrow N$ be the universal covering map, and let $\hat{N} \rightarrow N$ be the covering map associated to H , that is, $\pi_1(\hat{N}) = H$. Let $\tilde{\mathcal{F}}$ and $\hat{\mathcal{F}}$ be the foliations of \tilde{N} and \hat{N} induced by \mathcal{F} , both are foliations by planes. It follows from the last part of Theorem 4 that $\tilde{N} = \mathbb{R}^n$. Then the Euler characteristics of \tilde{N} and N satisfy

$$1 = \chi(\mathbb{R}^n) = \chi(\tilde{N}) = k\chi(\hat{N}), \quad \chi(\hat{N}) \in \mathbb{Z}.$$

Hence $k = 1$ and H is trivial.

Finally, if $\pi_1(N)$ is finite we obtain that $\pi_1(N)$ is trivial, consequently, by the last part of Theorem 4 we have that N is diffeomorphic to \mathbb{R}^n . \square

Proof of Theorem 1. We have the following possibilities

- (a) *N is orientable and \mathcal{F} transversally orientable:* by Palmeira's theorem, $\pi_1(N)$ is free.

- (b) *N is orientable and \mathcal{F} is not transversally orientable*: we consider $\pi : \tilde{N} \rightarrow N$ the double covering map that turn $\tilde{\mathcal{F}}$, the lifting to \tilde{N} of \mathcal{F} , transversally orientable. Then, by item (a) and Proposition 1, we have respectively that $\pi_1(\tilde{N})$ is free and torsion-free. Finally, since the index of $\pi_*(\pi_1(\tilde{N}))$ in $\pi_1(N)$ is equal to two, it follows from Lemma 1 that $\pi_1(N)$ is free.
- (c) *N non-orientable*: let $\pi : \tilde{N} \rightarrow N$ be the double covering map that turn \tilde{N} orientable. By applying the items (a) and (b) to \tilde{N} and $\tilde{\mathcal{F}}$, the lifting to \tilde{N} of \mathcal{F} , we obtain that $\pi_1(\tilde{N})$ is free. As above, since the index of $\pi_*(\pi_1(\tilde{N}))$ in $\pi_1(N)$ is equal to two, it follows from Lemma 1 that $\pi_1(N)$ is free.

□

Let X be a finite set and denote by $\#X$ the cardinality of X . Let G be a torsion-free finitely generated group. The *rank* of G is given by

$$\text{rank}(G) = \min\{k \in \mathbb{N}; k = \#X \text{ where } G \text{ is generated by } X\}$$

The following result is equivalent to Corollary 1.

Theorem 5. *Let \mathcal{F} be a foliation by planes of an open connected n -manifold N . If every leaf of \mathcal{F} is closed, then each finitely generate Abelian subgroup H of $\pi_1(N)$ has rank at most one.*

Proof. Let $p : \tilde{N} \rightarrow N$ be the covering such that $p_*(\pi_1(\tilde{N})) = H$. Then by Theorem 4, \tilde{N} is diffeomorphic to $\Sigma^2 \times \mathbb{R}^{n-2}$, where Σ^2 is an open surface. Consequently, since $H = p_*\pi_1(\tilde{N}) = p_*\pi_1(\Sigma^2)$ is an Abelian group, we have that Σ^2 is either an open cylinder or a plane. This completes the proof. □

Corollary 3. *If each leaf is closed and $\pi_1(N)$ is not Abelian, then $C = Z(\pi_1(N))$, the center of $\pi_1(N)$, is trivial.*

Proof. Suppose by contradiction that $C \neq \{1\}$. Since $\pi_1(N)$ is not Abelian, it follows that $C \subsetneq \pi_1(N)$. Let $\alpha \in \pi_1(N) \setminus C$ and H the Abelian subgroup of $\pi_1(N)$ which is generated by $\{\alpha\} \cup C$. Then, by Theorem 1, $\text{rank}(H) > 1$ contradicting Theorem 5. □

Remark 1. *Note that, if $\pi_1(N)$ has an Abelian subgroup of rank greater to one, or $C \neq \{1\}$, then N cannot be foliated by closed planes.*

4. CLASSIFYING OPEN 3-MANIFOLDS FOLIATED BY PLANES WHOSE FUNDAMENTAL GROUP IS ABELIAN OF RANK AT LEAST TWO

In this section we prove the Theorems 2 and 3. We state some results in 3-manifolds topology and foliations by planes which will be used in the proof of the Theorems 2 and 3. We begin with some terminology. A three manifold N is called *irreducible* if every embedded two sphere in N bounds a three ball in N . A two sided surface S in N is *incompressible* if the map $\pi_1(S) \rightarrow \pi_1(N)$ induced by the inclusion is injective. The following results are well-know tools in three manifold topology.

Theorem 6. [10, Theorem 6] *Let N be an orientable 3-manifold, not necessarily compact, and \mathcal{F} be a foliation of N by planes. Then N is irreducible.*

Gabai, in [3, Proof of Corollary 8.6], proved the following result.

Theorem 7. *If N is an open orientable irreducible 3-manifold and $\mathbb{Z} \times \mathbb{Z}$ is a subgroup of $\pi_1(N)$, then N contains an incompressible torus.*

4.1. Proof of Theorem 2. The following result is crucial for to show the Theorem 2.

Proposition 2. *Let \mathcal{F} be a foliation by planes of an open connected n -manifold N . If $H \cong \mathbb{Z}^k$ is a subgroup of $\pi_1(N)$, then $k < n$.*

To prove this proposition we need the following classical result in algebraic topology, see for example [5].

Lemma 2. *Let X be a normal space, F a closed subspace of X and Y an ENR space. Suppose that $f_0 : F \rightarrow Y$ and $g : X \rightarrow Y$ are continuous maps. If f_0 is homotopic to $g_0 = g|_F$, then there exists a continuous map $f : X \rightarrow Y$ such that $f|_F = f_0$ and f is homotopic to g .*

Let X be a topological space and G a group. We say that X is a $K(G, 1)$ -space if $\pi_1(X) = G$ and $\pi_p(X) = \{1\}, p \geq 2$.

Proof of Proposition 2. Firstly we claim that

(a) N is a $K(\pi_1(N), 1)$ -space.

In fact, let \tilde{N} be universal covering of N . It follows, by Theorem 4, that $\tilde{N} = \mathbb{R}^n$. Consequently, $\pi_p(\tilde{N}) = \{1\}$, for all $p = 1, 2, \dots, n$. Consider the exact sequence of homotopy groups

$$\cdots \rightarrow \pi_p(F) \rightarrow \pi_p(\tilde{N}) \rightarrow \pi_p(N) \rightarrow \pi_{p-1}(F) \rightarrow \cdots$$

where $F = \pi_1(N)$ (the fiber). Since $\pi_p(F) = \{1\}$, $p \geq 1$, we obtain that $\pi_p(\tilde{N}) = \pi_p(N)$ for all $p \geq 1$. This proves our claim.

Now, we have two possibilities.

(b) *Case $H = \pi_1(N)$:*

By contradiction, we suppose that $\pi_1(N) \cong \mathbb{Z}^n$. Since N and \mathbb{T}^n are $K(\mathbb{Z}^n, 1)$ -spaces, we obtain that N and \mathbb{T}^n have the same homotopic type. Consequently, by a classical theorem of algebraic topology (Whitehead's Theorem), $H_p(N) = H_p(\mathbb{T}^n)$ for all $p = 1, \dots, n$. But, as N is open and connected, $H_n(N) = 0$. This contradicts the fact that $H_n(N) = H_n(\mathbb{T}^n) = \mathbb{Z}$ and, therefore proves that $k < n$.

(c) *General case:*

Let us consider $p : \hat{N} \rightarrow N$ be the covering map associated to H , that is, $\pi_1(\hat{N}) = H$. Let $\hat{\mathcal{F}}$ be the foliations of \hat{N} induced by \mathcal{F} , this foliation is by planes. Thus, by the first case we obtain $k < n$. \square

Proof of Theorem 2. By Proposition 2, every Abelian subgroup of $\pi_1(N)$ is isomorphic to \mathbb{Z} or to \mathbb{Z}^2 . Consequently the fundamental group of N is a free product of the form

$$\pi_1(N) = H_1 * \dots * H_\ell,$$

where H_j , $j = 1, \dots, \ell$, are subgroups of $\pi_1(N)$ isomorphic to \mathbb{Z} or \mathbb{Z}^2 , respectively.

We suppose that $j \in \{1, \dots, \ell\}$ is such that H_j is isomorphic to \mathbb{Z}^2 . By Theorems 6 and 7, there exists an incompressible 2-torus T_j contained in N such that $\pi_1(T_j) = H_j$. Deforming T_j , if necessary, by an isotopy of identity we can assume that T_j is in general position with respect to \mathcal{F} . Then, \mathcal{F} induces a foliation \mathcal{G}_j on T_j having finitely many singularities each of which is locally topologically equivalent either to a center or to a saddle point of a vector field. We can assume that the foliation \mathcal{G}_j is defined by a vector field G_j . Certainly, we may assume that T_j has been chosen so that no pair of singularities of G_j is into the same leaf of \mathcal{F} , in other words, G_j has no saddle connections.

(a) *G_j is topologically equivalent to irrational flow.*

In fact, by using Rosenberg's arguments (see [10, pag. 137]), we may deform T_i by an isotopy of identity so that G_j has no singularities. Hence, since \mathcal{F} is by planes and T_j is incompressible, G_j has no closed orbit. Therefore, G_j is topologically equivalent to irrational flow.

Let N_j be the saturated by \mathcal{F} of T_j . Clearly, N_j is a open 3-submanifold of N invariant by \mathcal{F}). We will show that N_j is homeomorphic to $\mathbb{T}^2 \times \mathbb{R}$. Let \mathcal{F}_j be the restriction of \mathcal{F} at N_j and $p : T\mathcal{F}_j \rightarrow N_j$ be the vector bundle such that for all $x \in N_j$ the fiber $p^{-1}(x)$ is tangent at x to leaf of \mathcal{F}_j (which is also a leaf of \mathcal{F}) passing by x . By deforming T_j , if necessary, we can assume that:

- (b) *If $\theta : T_j \rightarrow TN_j$ is the normal vector field to T_j in N_j , then θ is tangent to \mathcal{F} , that is, $\theta : T_j \rightarrow p^{-1}(T_j)$.*

By (a) of the proof of Proposition 2, we have that N_j is a $K(\pi_1(N_j), 1)$ -space. Then, by Whitehead's theorem, the inclusion $i : T_j \rightarrow N_j$ is a homotopy equivalence. Consequently:

- (c) *there exists a continuous map $f : N_j \rightarrow T_j$ such that $f \circ i$ is homotopic to id_{T_j} and $i \circ f$ is homotopic to id_{N_j}*

We claim that

- (d) *there exists a retraction $r : N_j \rightarrow T_j$ such that $r|_{T_j} = id_{T_j}$ and r is homotopic to f .*

Indeed, calling $X = N_j = Y$, $F = T_j$, $f_0 = f|_{T_j}$ e $g = f : N_j \rightarrow N_j$, by Lemma 2, there exists $r : N_j \rightarrow T_j$ such that $r|_{T_j} = id_{T_j}$ and r is homotopic to f .

- (e) *N_j is homeomorphic to $\mathbb{T}^2 \times \mathbb{R}$.*

The pull-back of the vector bundle $p^{-1}(T_j) \rightarrow T_j$ by the retraction $r : N_j \rightarrow T_j$ given in (d) is exactly the vector bundle $T\mathcal{F}_j \rightarrow N_j$ since $r^*(p^{-1}(T_j)) = T\mathcal{F}_j$. Consequently, the vector field $\Theta = r^*\theta$ is tangent to \mathcal{F}_j and is normal to T_j . This implies that N_j is homeomorphic to $\mathbb{T}^2 \times \mathbb{R}$.

Finally, if for $i \neq j$ the group H_i is isomorphic to \mathbb{Z}^2 , then we obtain another open 3-submanifold of N which is invariant by \mathcal{F} and homeomorphic to $\mathbb{T}^2 \times \mathbb{R}$. Furthermore, $N_i \cap N_j = \emptyset$. \square

4.2. Proof of Theorem 3.

Proof of Theorem 3. By Theorem 2, there exists N_0 an open submanifold of N homeomorphic to $\mathbb{T}^2 \times \mathbb{R}$ which is invariant by \mathcal{F} such that $(\mathcal{F}|_{N_0}, N_0)$ is topologically equivalent to $(\mathcal{F}_0 \times \mathbb{R}, \mathbb{T}^2 \times \mathbb{R})$ where \mathcal{F}_0 is the foliation on \mathbb{T}^2 defined by the irrational flow. Note that $\pi_1(N_0) = \pi_1(N) \cong \mathbb{Z}^2$

Let B be a connected component of $N \setminus \overline{N_0}$. Then

- (a) *B is homeomorphic to \mathbb{R}^3 and every leaf in B is closed.*

Firstly we claim that $\pi_1(B) = \{1\}$. In fact, by using the Van Kampen Theorem, we have that $\pi_1(B) * \pi_1(N_0)$ is subgroup of $\pi_1(N)$ and hence, since $\pi_1(N_0) = \pi_1(N)$, we have that

$\pi_1(B) = \{1\}$. Now, we are going to show that every leaf in B is closed. We suppose, by contradiction, that there exists a non closed leaf. By classical arguments in the theory of foliations, there exists a simple closed curve $\gamma \subset B$ which is transversal to \mathcal{F} . Moreover, γ is not nullhomotopic in B , otherwise, by using Heafliger's arguments, we obtain a loop α in a leaf of \mathcal{F} which is non trivial in this leaf contradicting the fact that all the leaves are planes. Therefore γ is not nullhomotopic in B , but this contradicts that $\pi_1(B) = \{1\}$. Thus every leaf in B is closed. Thence, by Palmeira's Theorem, we have that B is homeomorphic to \mathbb{R}^3 and $(\mathcal{F}|_B, B)$ is topologically equivalent to $(\mathcal{F}_b \times \mathbb{R}, \mathbb{R}^2 \times \mathbb{R})$ where \mathcal{F}_b is an foliation by lines on \mathbb{R}^2 .

Finally by using the Van Kampen Theorem it follows that every leaf in B separates N . \square

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